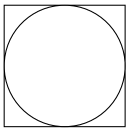
901. A student tries to find the intersection of the graphs $x^2 + y^2 = 1$ and $5x + 6y = 8$, and receives an error message from her calculator. Show that this is correct, and explain the result.

902. Determine the value of $\ln \sqrt{e} + \ln \sqrt[3]{e} + \ln \sqrt[6]{e}$.

903. Show that the best linear approximation to the function $\text{f}(x) = 3x^3 - 5x$, for x values close to 1, is given by $\text{f}(x) \approx 4x - 6$.

904. A square has a circle inscribed.



A point is then chosen at random in the square. Show that the probability that the point is inside the circle is $\frac{\pi}{4}$.

905. State, with a reason, whether variables related in the following ways are likely to display correlation if a sample of values is taken. In each case, $\bar{x} = 0$.

(a) $x + y$ is approximately constant,

(b) $x^2 + y$ is approximately constant,

(c) $x^3 + y$ is approximately constant.

906. A quadratic sequence Q_n begins $1, 8, 21, \ldots$. By considering second differences, or otherwise, find an nth term formula.

907. Prove that the gradients of perpendicular lines are negative reciprocals.

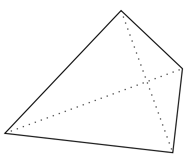
908. A parabola has equation $x = 2y^2 + 1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{1}{4y}$.

(b) Hence, show that the tangent at $y = \frac{1}{8}$ has equation $32x = 16y + 31$.

909. On a set of Cartesian axes, sketch the locus of points which satisfy $(x - 2)(y - 3) = 0$.

910. A kite has diagonals of length a and b:



Prove that the area of the kite is $A = \frac{1}{2}ab$.

911. Find the double root of the following equation:

$$\left(x^3 + a^3\right)\left(x^2 - a^2\right) = 0.$$

912. The solution of the following equation is the same for all but one value of the constant b.

$$\frac{(x^2 + 2)(x^2 - 3)}{x^2 + b} = 0.$$

Write down that value.

913. The interior angles of a quadrilateral are in AP. Give, in radians, the set of possible values for the smallest angle.

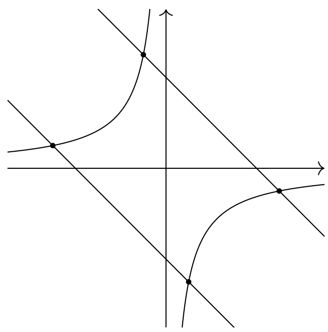
914. Kepler's third law of planetary motion states that the orbital period T and radius r of a planetary orbit are related by $T^2 \propto r^3$.

(a) Assuming that the Earth orbits the sun once every 365 days at a distance of 152 million km, find r in terms of T, using units of millions of km and years.

(b) Hence, determine the radius of the orbit of an asteroid whose orbital period is 2 years.

915. If $\log_9 y = x$, write 3^x as a function of y.

916. Collectively, the graphs $x + y = 4$, $x + y = -4$ and $xy = -5$ meet at a total of four points.



Prove that these points form a rectangle.

917. A mathematician sets up a function to answer the following question: "How many real roots does the cubic equation $ax^3 + bx^2 + cx + d = 0$ have?" Write down

(a) the largest possible domain,

(b) a suitable codomain,

(c) the range, with the domain given.

918. Sketch $y = \sqrt[3]{x}$.

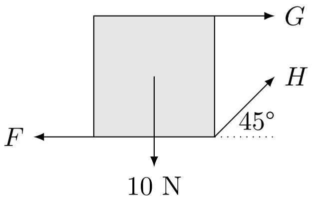
919. A four-sided die and a six-sided die are rolled at the same time. Find the probability that the scores on the two dice are the same.

920. Simplify the following expression:

$$\frac{(2^x + 1)^3 + (2^x - 1)^3}{2^x}.$$

921. Two polynomial curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are drawn, where f has degree m and g has degree n. Prove that the greatest possible number of points of intersection is $\max(m, n)$.

922. In the following force diagram of a square object in equilibrium, find the force magnitudes F, G, H.



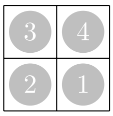
923. Show that the line $y = 456x - 3600$ is a tangent to the curve $y = x^3 + x^2$.

924. By first squaring the equations, find all possible values of R satisfying $R\sin\theta = 6$ and $R\cos\theta = 8$.

925. A particle has initial velocity $5\mathbf{i} + 3\mathbf{j}$ ms^{-1}, and constant acceleration $2\mathbf{i} - 4\mathbf{j}$ ms^{-2}. Determine the average velocity over the first five seconds.

926. Sketch the graphs $y = e^x$ and $y = -e^{-x}$.

927. A set of counters are numbered $1, 2, 3, 4$. They are placed at random in a two-by-two grid.



Find the probability that, when reading clockwise around the grid, the counters appear in ascending order. The diagram shows a successful outcome.

928. Show that the lines $x + y = 1$ and $y - 4x = 7$ intersect inside the circle $x^2 + 2x + y^2 - 3y + 2 = 0$.

929. In each case, describe the symmetry of the graph $y = \mathrm{f}(x)$, when f has the given property.

(a) $\mathrm{f}(-x) \equiv \mathrm{f}(x)$,

(b) $\mathrm{f}(-x) \equiv -\mathrm{f}(x)$.

930. Complete the square in $5a^4b^2 + 100a^2b + 1$.

931. Two cards are dealt from a deck, with replacement. State which, if either, of the following events has the greater probability:

① a jack, then another jack,

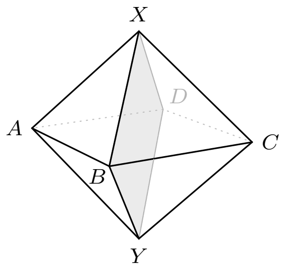
② a king, then a queen.

932. Prove that the area of an equilateral triangle may be expressed, in terms of its perimeter, as

$$A = \frac{\sqrt{3}P^2}{36}.$$

933. Two hikers leave camp simultaneously. A walks on bearing 90° at 3 mph; B walks on bearing 300° at 3.5 mph. Determine the bearing of B from A once they have left camp, and show that it is constant.

934. A regular octahedron is shown in the diagram. A quadrilateral $BXDY$ has been shaded:



State, in the most precise terms possible, what type of quadrilateral $BXDY$ is.

935. Express $3t^2 + 4t - 6$ as a quadratic in $(t - 1)$.

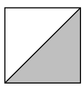
936. State, with a reason, whether the following holds: "Friction and reaction, if both act at the same point on the same object, are perpendicular."

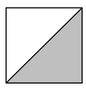
937. The curve $y = \sqrt{x} - 2$ crosses the x axis at $x = 4$. Determine the area enclosed by the curve, the x axis and the y axis.

938. In this question, f is a polynomial function.

Prove that, if $\mathrm{f}'(x)$ is given but $\mathrm{f}(x)$ is not, then it is possible to calculate $\mathrm{f}(p) - \mathrm{f}(q)$ for any $p, q \in \mathbb{R}$, but it is not possible to calculate $\mathrm{f}(p) + \mathrm{f}(q)$.

939. The two tiles depicted below are placed side by side, each in a random orientation, such that they share an edge.





Find the probability that the shading forms one contiguous region.

940. In each case, a list of quantities is given, which refers to a finite arithmetic series. State whether knowing the relevant quantities would allow you to evaluate the sum of the series, giving a reason if your answer is "No."

(a) Number of terms; mean.

(b) First term; last term; mean.

(c) Number of terms; first term; last term.

941. Find the angle, in radians, between the hands of a clock at twenty to six.

942. Solve the equation $2n^2(2n-1)-(2n-1)^2=0$.

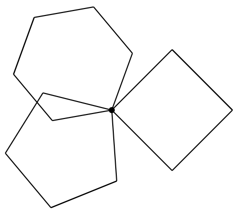
943. Simplify the following, for an invertible function f:

(a) $\mathrm{f}^{-1}\mathrm{f}\,\mathrm{f}^{-1}(x)$,

(b) $\mathrm{f}^{-1}\mathrm{f}^2(x)$.

944. A student writes: "Friction always acts to oppose motion. So, when a car is not moving, there can be no friction acting on it." Explain carefully why this statement is incorrect.

945. A regular hexagon, a regular pentagon and a square share a vertex V.



Show that at least 42° of angle around V is exterior to all three shapes.

946. The definite integral below gives the displacement, over a particular time period, for an object moving with constant acceleration:

$$s=\int_2^6 1+\tfrac{2}{5}t\,dt.$$

(a) Write down the acceleration and duration.

(b) Find the initial and final velocities.

(c) Calculate the displacement by integration.

(d) Verify the answer to part (c) using a constant acceleration formula.

947. Find the length of the line segment

$$x=3+6\lambda,\quad y=-1+8\lambda,\quad \lambda\in[-1,1].$$

948. Two six-sided dice are rolled, and the scores are summed. A student suggests that the total could be modelled with $X\sim B(12,{}^1\!/_6)$. Explain how you know that this is incorrect.

949. Forces $\mathbf{F}=a\mathbf{i}+6b\mathbf{j}$ and $\mathbf{G}=(2b+1)\mathbf{i}+(4+3a)\mathbf{j}$ act on an object. Show that, unless other forces act, the object cannot remain in equilibrium.

950. Shade the region of the (x,y) plane which satisfies both of the following inequalities:

$$|x|<2,\qquad y\geq 0.$$

951. State true or false for the following:

(a) "Every square is a rectangle."

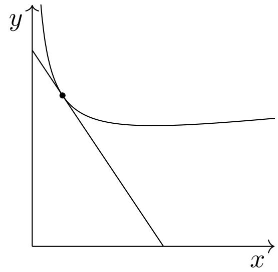
(b) "Not all rectangles are parallelograms."

(c) "Some kites are trapezia."

952. Given $\mathrm{f}(x)=x-1$, solve for a in

$$\int_0^{\mathrm{f}(a)}\mathrm{f}(x)\,dx=0.$$

953. The curve $y=x^{\frac{1}{2}}+4x^{-\frac{1}{2}}$ has a tangent drawn to it at $x=1$.



Show that this tangent has equation $3x+2y=13$.

954. Write the following in simplified interval notation:

$$\{x\in\mathbb{R}:-1<x<3\}\cap\{x\in\mathbb{R}:|x+2|>2\}.$$

955. State, with a reason, whether $y=x^3$ intersects the following curves:

(a) $y=x^2+1$,

(b) $y=x^3+1$,

(c) $y=x^4+1$.

956. Show that the largest angle in a triangle with sides $(10,12,15)$ is approximately 1.5 radians.

957. Two fair coins are tossed together. Given that at least one head is observed, find the probability that two heads are observed.

958. Sketch the following graphs, where $a<b$,

(a) $x=(y-a)(y-b)$,

(b) $x=(y-a)^2(y-b)$.

959. Two vectors $\mathbf{a}$ and $\mathbf{b}$ can be expressed in terms of the standard perpendicular unit vectors as

$$\mathbf{a}=4\mathbf{i}+3\mathbf{j},$$
$$\mathbf{b}=2\mathbf{i}-5\mathbf{j}.$$

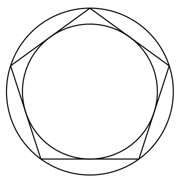
Express $16\mathbf{i}+25\mathbf{j}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

960. Find the equation of the normal to the curve $(x-1)^2+y^2=25$ at the point $(4,4)$, giving your answer in the form $ax+by+c=0$, for $a,b,c\in\mathbb{Z}$.

961. Prove that, if a sequence is quadratic with nth term $u_n = an^2 + bn + c$, then the second difference of the sequence is constant, with value $2a$.

962. Show that $\mathrm{g}(x) = x^2 + 4x + 2$ has two fixed points.

963. In a regular polygon, the *apothem* a is the radius of the largest circle which can be inscribed in the polygon. The *circumradius* R is the radius of the smallest circle within which the polygon can be inscribed.



Determine the ratio $a : R$ for

(a) an equilateral triangle,
(b) a square,
(c) a regular hexagon.

964. The curve $y = \mathrm{f}(x)$ is to be differentiated from first principles.

(a) Explain the significance of the numerator and denominator in the expression

$$\lim_{h\to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}.$$

(b) Explain why the same result would be attained by using the expression

$$\lim_{h\to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x-h)}{2h}.$$

965. The inequality $x^2 + px + q > 0$ has solution set $(\infty, 4) \cup (5, \infty)$. Find p and q.

966. Show that ${}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 \equiv \dfrac{n^2 + 3n}{2}$.

967. The graph $y = a(x+b)^2(x+c)^3 = 0$ touches the x axis at $x = 1$, crosses the x axis at $x = -4$, and crosses the y axis at $y = 16$. Find the values of the constants a, b and c.

968. For events A and B, write down the values of the following probabilities:

(a) $\mathbb{P}(A \mid A)$,
(b) $\mathbb{P}(A \cap B' \mid B)$,
(c) $\mathbb{P}(A \cap B \mid A' \cup B')$.

969. Show that the curve $y = x^6 + x^3 + 1$ does not cross the x axis.

970. A function f is defined over the reals, and has range $[-a, a]$. Give the ranges of the following:

(a) $x \mapsto \mathrm{f}(x) + a$,
(b) $x \mapsto \mathrm{f}(x) - a$,
(c) $x \mapsto a - \mathrm{f}(x)$.

971. Show that $\displaystyle\int_2^4 \frac{3(2x+1)^3 - 3}{x}\, dx = 700$.

972. At the point with x coordinate a, the tangent line to $xy = 1$ has equation

$$y = -\frac{1}{a^2}x + c.$$

(a) Explain the coefficient $-\dfrac{1}{a^2}$.
(b) By substituting, show that a general tangent line to the curve $xy = 1$ has equation

$$y = -\frac{1}{a^2}x + \frac{2}{a}.$$

973. Solve the following equation:

$$\sum_{i=0}^{2} (1 + 4i - 3i^2)x^i = 0.$$

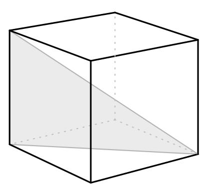
974. The variables x and y are defined, in terms of the variables a and b, by

$$x = \sqrt{3}a - b,$$
$$y = a + \sqrt{3}b.$$

Express a and b in terms of x and y.

975. Determine the least n such that the product of n consecutive integers necessarily ends in 0.

976. The diagram shows a cube of unit side length.



Find angle ACB, in the form $\arcsin k$.

977. Find the unknown constant p, if the following may be expressed as a linear function of x:

$$\frac{x^2 + px + 2}{x - 6}.$$

978. A child mounts a small fan on the back of a toy sailing boat, and sets it to blow air forwards into the sail. Describe, with reference to Newton's laws, what will happen when the fan is turned on.

979. Sketch $\sqrt{y} = x - 1$.

980. Find the probability that, when two dice are rolled, the scores differ by less than two.

981. Two functions f and g are such that the indefinite integral of their sum is quadratic. Determine the number of roots of the equation $\mathrm{f}(x) + \mathrm{g}(x) = 0$.

982. Prove or disprove the following statement:

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \iff \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A \cap B).$$

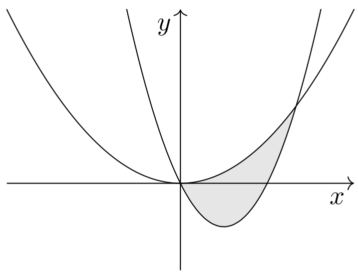
983. State, with a reason, whether the following gives a well-defined function:

$$\mathrm{h}: \begin{cases} [0,1] \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{(x-2)(x-3)}. \end{cases}$$

984. An arithmetic progression has common difference 4, and the product of its first and third term is -7. Find all possible values of the second term.

985. Prove that the difference between two consecutive odd squares is divisible by 8.

986. Region R of the (x, y) plane, which is shaded in the diagram below, is defined as those points which satisfy $4x^2 - 2x \le y \le x^2$.



The area of R is calculated with

$$A_R = \int_0^k 2x - 3x^2 \, dx.$$

(a) Find k.

(b) Explain the form of the integrand $2x - 3x^2$.

(c) Hence, find the area of region R.

987. On a standard 8×8 chessboard, find an expression for the number of ways of placing

(a) 8 identical pawns,

(b) 8 identical white and 8 identical black pawns.

988. Show that, if vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ are perpendicular, then $a_1 b_1 + a_2 b_2 = 0$.

989. Describe the two transformations which take the graph $y = \mathrm{f}(x)$ onto the graph $y = 2\,\mathrm{f}(2x)$.

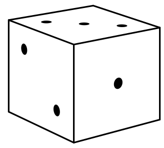
990. S is the set of (x, y) points which simultaneously satisfy both of the following equations:

$$2x + 3y = 7,$$
$$4x = 1 - 6y.$$

Find $\mathrm{n}(S)$, the number of elements in S.

991. Simplify $(k - 4, k + 3] \cap (k - 1, k + 6]$.

992. Opposite faces of a six-sided die add up to seven.



Prove that, with this restriction, there are only two different ways of arranging the faces on a die.

993. Find the area of a triangle with sides $(4, 5, 6)$, giving your answer as a surd.

994. A packing case of mass 20 kg is being hauled up a rough ramp of inclination 7° with a light rope and a winch. The coefficient of friction is 0.215, and the winch moves the packing case at a constant speed of 0.15 ms^{-1}. Find the tension in the rope.

995. The monic parabola $y = x^2 + 2x + 3$ has a minimum at point (p, q). Find the equation of the monic parabola which has a minimum at point $(p, -q)$.

996. Variables x and y take the following values:

x	0	1	2	3
y	0	2	6	10

Show that the relationship cannot be quadratic.

997. A set of n letters, all different, has 40320 anagrams. Find n.

998. A student is attempting to find constants A, B which will make the following identity hold:

$$\frac{1}{x^3 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x - 1}.$$

(a) Show that this can be written as

$$1 \equiv Bx^2 + Ax - A.$$

(b) Show that there are no constants A, B which make this an identity.

(c) Suggest an alteration to the RHS which would allow an identity.

999. Determine which of the following sequences attains the value 1000 in the fewest iterations:

$$a_1 = 1, \quad a_{n+1} = 1.01 \times a_n,$$
$$b_1 = 1, \quad b_{n+1} = b_n + 1.$$

1000. Solve the equation $2x^{0.4} + x^{0.2} = 1$.

——— End of Volume I ———